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HEAT CONDUCTION IN SPHERES PACKED IN AN INFINITE REGULAR CUBICAL ARRAY

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Abstract—Steady-state temperature solutions are sought for an infinite cubical array of spheres. Heat transfer is by conduction and constant properties are assumed. The problem is treated as a unit cube containing a sphere at the center. Due to symmetry the cube is further subdivided into a wedge representing the unit cell. Applying continuity and boundary conditions, the analytical temperature solutions are obtained. For the case in which the sphere is assumed to represent porosity in a solid, a porosity correction to thermal conductivity is obtained in the form:

$$f(p) = e^{-2.14p} \quad 0.0 < p < 0.30$$
$$= 0.92 - 1.34p \quad 0.30 < p < 0.50$$

where p = fractional porosity

$$f(p) = \text{porosity correction factor}, f(p) \leq 1.$$

NOMENCLATURE

A_{00}^{nm}, $A_{\alpha\beta}^{nm}$, $A1_{\gamma\eta}$, $A2_{\gamma\eta}^{nm}$, $A3_{\gamma\eta}^{nm}$, coefficients of the spherical harmonics of components of the outer boundary condition;
$C_{00}^{(1)}$, $C_{nm}^{(1)}$, coefficients of the series for the temperature solution in the sphere;
$C_{00}^{(2)}$, $C_{nm}^{(2)}$, $C_{00}^{(3)}$, $C_{nm}^{(3)}$, coefficients of the series for the temperature solution in the cube;
$E(\rho)$, $F(\mu)$, $G(\phi)$, eigenfunction solution to Laplace's equation in the spherical coordinate system;
$\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_z$, unit vectors in the x- and z-directions;
$\hat{\mathbf{e}}_\rho$, $\hat{\mathbf{e}}_\alpha$, $\hat{\mathbf{e}}_\phi$, unit vectors in the ρ-, α- and ϕ-directions;
$f(p)$, porosity correction factor for fractional porosity p;
$H^{(1)}(\mu, \phi)$, $H_{nm}^{(2)}(\mu, \phi)$, $H_{nm}^{(3)}(\mu, \phi)$, $H^{(4)}(\mu, \phi)$, components of the outer-boundary condition defined for the top, outer side, and bottom faces of the unit cell;
k_s, k_c, conductivities of the sphere and cube materials of the unit cell;
k_p, conductivity of a porous material with fractional porosity p;
k_{100}, conductivity of 100% dense material;
k, l, m, n, summation indices;
$\hat{\mathbf{n}}_l$, unit vector normal to the side face;
$p_n^m(\mu)$, associated Legendre functions of degree n and order m;
$\mathbf{q}''$, heat flux vector;
q_T'', heat flux crossing the top face;
$\overline{q_T''}$, average heat flux on the top face;
R_c, half length of a side of the cubic unit cell;
R_s, radius of the sphere;
R_a, non-dimensional sphere radius ($= R_s/R_c$);
S, pitch between centers of cubic unit cell ($2R_c = S$);
$T(\rho, \mu, \phi)$, non-dimensional temperature;
$T_s(\rho, \mu, \phi)$, non-dimensional temperature field within the sphere;
$T_c(\rho, \mu, \phi)$, non-dimensional temperature field within the cube but outside the sphere;
T_T, T_B, non-dimensional boundary temperatures on the top and bottom faces, respectively;
x, y, z, independent variables for a rectangular coordinate system with origin at the sphere center;
ρ, α, ϕ, independent variables for a spherical coordinate system with origin at the sphere center;
μ, $= \cos\alpha$;
β, γ, summation indices;
∇, gradient operator;
∇^2, laplacian operator.

1. INTRODUCTION

SOLID materials commonly contain inclusions of a second dissimilar phase. When a temperature gradient is applied, these inclusions perturb the flow of heat through the material. If the inclusions are gas filled pores, they may significantly alter the thermal performance.

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In this study the conduction of heat through a material with an idealized arrangement of inclusions is examined. It is assumed that the inclusions are spherical and ordered in a regular simple cubical array throughout the medium. Constant thermal properties are assigned to both the spheres and the remaining solid. Explicit analytical expressions are sought for the temperature distribution within a characteristic unit cell. A correction factor for the thermal conductivity of the solid, which accounts for the presence of the porosity, is then found from the solutions. This correction formula is compared with other expressions found in the literature.

2. TEMPERATURE SOLUTIONS

2.1. *Unit cell and boundary conditions*

If the heat flow through the medium is normal to the pitch between sphere centers of adjacent nearest neighbors, it is possible to consider a single plane of spheres, shown in Fig. 1(a), which can be reduced further to a cube containing a single sphere, Fig. 1(b). Due to symmetry the side faces of the cube are adiabatic and the top and bottom faces are taken to be known uniform temperatures.

The governing equations are written in terms of non-dimensional quantities for temperature and position. The temperature is unity on the top boundary and zero on the bottom boundary.

Due to symmetry the cube and sphere can be subdivided further into eight 45° wedges [Fig. 1(c)]. The top and bottom faces retain the boundary conditions T_t and T_b while all side faces are adiabatic. In this unit cell, as shown in Fig. 2, spherical and rectangular coordinate systems are located at the center. In the solutions discussed below, the term "cube" will be used to designate the region in the unit cell outside the sphere.

The applicable boundary conditions for the problem are:

finite temperature at all points;

continuity of temperature at the sphere–cube interface;

continuity of heat flux at the sphere–cube interface;

all three side-faces are adiabatic;

The conditions on the top, bottom and outer side faces are recognized to be three components to the same boundary condition.

The adiabatic boundary condition requires that the component of the heat flux normal to the surface be zero, and may be stated as

$$\hat{\mathbf{n}}_i \cdot \mathbf{q}''(\rho, \alpha, \phi)\Big|_{S_i} = 0 \qquad (1)$$

where $\hat{\mathbf{n}}_i$ is the normal to the ith surface, S_i, and $\mathbf{q}''$ (ρ, α, ϕ) is the heat flux vector.

2.2. *General form of temperature solution*

For constant properties and no internal heat source, steady-state heat conduction within the unit cell is governed by Laplace's equation

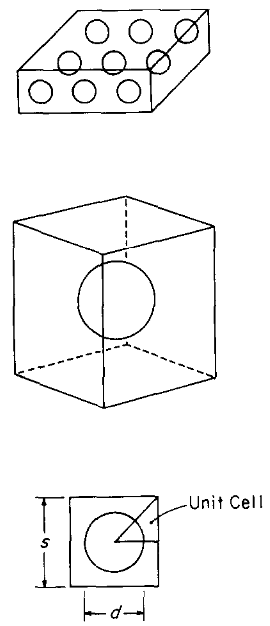

FIG. 1. Development of the unit cell: (a) repeating horizontal plane of spheres; (b) single sphere centered within a cube; (c) central sphere in an array with the unit cell defined.

$$\nabla^2 T = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial T}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial T}{\partial \mu}\right] + \frac{1}{\rho^2(1-\mu^2)}\frac{\partial^2 T}{\partial \phi^2} = 0 \qquad (2)$$

where $\mu = \cos\alpha$ (Fig. 2), T and ρ are non-dimensional temperatures and positions in the unit cell. The normalized dimensions are chosen such that the half length of a cube side, or pitch, is unity; i.e. $S/2 = 1$.

The solution to equation (2) may be represented term by term with an eigenfunction solution of the form [1]

$$T(\rho, \mu, \phi) = E(\rho)\,F(\mu)\,G(\phi). \qquad (3)$$

The specific solution for the temperature in the sphere may be obtained by requiring finite temperatures in the cell and specifying that the $\phi = 0$ and $\phi = \pi/4$ faces are adiabatic.

$$T_s(\rho, \mu, \phi) = C_{00}^{(1)} + \sum_{n=1}^{\infty}\sum_{m=0}^{[n/4]} \times C_{nm}^{(1)}\rho^n P_n^{4m}(\mu)\cos(4m\phi). \qquad (4)$$

In the cube, the solution is

$$T_c(\rho, \mu, \phi) = C_{00}^{(2)} + C_{00}^{(3)}/\rho + \sum_{n=1}^{\infty} \times \sum_{m=0}^{[n/4]} (C_{nm}^{(2)}\rho^n + C_{nm}^{(3)}\rho^{-n-1}) \times P_n^{4m}(\mu)\cos(4m\phi) \qquad (5)$$

where $[n/4]$ is the largest integer in $n/4$ and the C's are coefficients which must be determined through application of the remaining boundary conditions. $P_n^{4m}(\mu)$ is the associated Legendre function of degree n and order $4m$.

2.3. *Application of boundary conditions*

Since the temperature solution is expressed as a linear combination of eigenfunctions, the orthogonality property may be used to determine the coefficients. At the sphere–cube interface, $\rho = R_s/R_c = R_a$(R_s is the radius of the sphere), continuity of temperature from equations (4) and (5) can be expressed as

$$C_{00}^{(1)} + \sum_{n=1}^{\infty}\sum_{m=0}^{[n/4]} C_{nm}^{(1)} R_a^n P_n^{4m}(\mu)\cos(4m\phi)$$
$$= C_{00}^{(2)} + C_{00}^{(3)}/R_a + \sum_{n=1}^{\infty}\sum_{m=0}^{[n/4]} n(C_{nm}^{(2)} R_a^n + C_{nm}^{(3)} R_a^{-n-1})P_n^{4m}(\mu)\cos(4m\phi). \quad (6)$$

The boundary condition of continuity of heat flux at the sphere–cube interface, $\rho = R_a$, yields the expression

$$k_s \sum_{n=1}^{\infty}\sum_{m=0}^{[n/4]} n\, C_{nm}^{(1)} R_a^{n-1} P_n^{4m}(\mu)\cos(4m\phi)$$
$$= -k_c\{-C_{00}^{(3)}/R_a^2 + \sum_{n=1}^{\infty}\sum_{m=0}^{[n/4]} [nC_{nm}^{(2)} R_a^{n-1} - (n+1)\, C_{nm}^{(3)} R_a^{-n-2}]\, P_n^{4m}(\mu)\cos(4m\phi)\} \quad (7)$$

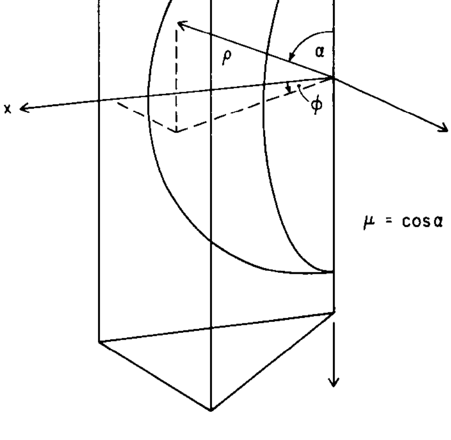

FIG. 2. Unit cell with coordinate system.

where k_s and k_c are the thermal conductivities of the sphere and cube materials, respectively.

Exploiting the orthogonality properties for the cosine and associated Legendre functions [2], equations (6) and (7) can be combined in a straightforward way to produce four algebraic equations which may be used in determining the arbitrary coefficients, $C^{(1)}$, $C^{(2)}$ and $C^{(3)}$

$$C_{00}^{(1)} = C_{00}^{(2)} + C_{00}^{(3)}/R_a \quad (8a)$$

$$C_{lk}^{(1)} R_a^l = C_{lk}^{(2)} R_a^l + C_{lk}^{(3)} R_a^{-l+1} \quad (8b)$$

$$C_{00}^{(3)} = 0 \therefore C_{00}^{(1)} = C_{00}^{(2)} \quad (8c)$$

$$k_s l C_{lk}^{(1)} R_a^{l-1} = k_c(lC_{lk}^{(2)} R_a^{l-1} - (l+1)C_{lk}^{(3)} R_a^{-l-2}). \quad (8d)$$

The remaining boundary condition is composed of three components, one each from the top face, the outer side face ($x = R_c$), and the bottom face. This yields the additional necessary equations to fully determine the coefficients in the temperature solutions. The normalized temperature on the top face is $T_s(\rho, \mu, \phi) = 1$. This is valid for the top face which is defined by the following relations

$$\rho = \frac{1}{\mu} \quad \text{for} \quad \frac{1}{(2+\tan^2\phi)^{1/2}} \le \mu \le 1 \quad \text{and} \quad 0 \le \phi \le \pi/4.$$

The boundary condition here can then be expressed as

$$1 = C_{00}^{(2)} + \sum_{n=1}^{\infty}\sum_{m=0}^{[n/4]} \left[C_{nm}^{(2)} \left(\frac{1}{\mu}\right)^n P_n^{4m}(\mu) \times \cos(4m\phi) + C_{nm}^{(3)} \left(\frac{1}{\mu}\right)^{-n-1} P_n^{4m}(\mu)\cos(4m\phi)\right]. \quad (9)$$

The side face, $x = R_c$, is adiabatic. This face is defined by the relationships

$$\rho = \frac{1}{\cos\phi(1-\mu^2)^{1/2}} \quad \text{for} \quad \frac{-1}{(2+\tan^2\phi)^{1/2}} \le \mu \le \frac{1}{(2+\tan^2\phi)^{1/2}} \quad \text{and } 0 \le \phi \le \pi/4.$$

The adiabatic condition on this face is expressed as

$$\hat{\mathbf{e}}_x \cdot \mathbf{q}'' = \hat{\mathbf{e}}_x \cdot [-k_c \nabla T_s(\rho, \mu, \phi)] = 0 \quad (10)$$

where $\hat{\mathbf{e}}_x$ is the unit vector normal to the side face.

To apply this condition, the definition of the gradient operator is inserted into equation (10). This yields

$$\hat{\mathbf{e}}_x \cdot \mathbf{q}'' = \hat{\mathbf{e}}_x \cdot \left\{ -k_c \left[\hat{\mathbf{e}}_\rho \frac{\partial}{\partial\rho} T_c(\rho, \mu, \phi) + \hat{\mathbf{e}}_\alpha \frac{1}{\rho}\frac{\partial}{\partial\alpha} T_c(\rho, \mu, \phi) + \hat{\mathbf{e}}_\phi \frac{1}{\rho\sin\alpha}\frac{\partial}{\partial\phi} T_c(\rho, \mu, \phi)\right]\right\}. \quad (11)$$

The dot products of the unit vectors are computed to be

$$\begin{aligned} \hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}}_\rho &= \sin\alpha \cdot \cos\phi \\ \hat{\mathbf{e}}_x \cdot \mathbf{e} &= \cos\alpha \cdot \cos\phi \\ \hat{\mathbf{e}}_x \cdot \mathbf{e}_\phi &= -\sin\phi. \end{aligned} \tag{12}$$

Recalling that $\mu = \cos\alpha$, the adiabatic condition on the side face may be written as

$$\begin{aligned} 0 = \sum_{n=1}^{\infty}\sum_{m=0}^{[n/4]} \Bigg\{ & \frac{C_{nm}^{(2)}}{\cos^{n-1}\phi(1-\mu^2)^{n/2}} \\ & \times \left[(nP_n^{4m}(\mu) - (n+4m)\mu P_{n-1}^{4m}(\mu) \sin\phi \right] \\ & \times \cos(4m\phi)\cos + 4mP_n^{4m}(\mu)\sin(4m\phi) \\ & + \frac{C_{nm}^{(3)}}{\cos^{-n-2}\phi(1-\mu^2)^{(-n-1/2)}} \\ & \times \left[((\mu^2(2n+1)-(n+1))P_n^{4m}(\mu) - (n+4m)\mu P_{n-1}^{4m}(\mu)) \right. \\ & \left. \times \cos(4m\phi)\cos\phi + 4mP_n^{4m}(\mu)\sin(4m\phi)\sin\phi \right] \Bigg\}. \end{aligned} \tag{13}$$

On the bottom face, the normalized temperature is $T_s(\rho, \mu, \phi) = 0$. The bottom plane is defined by

$$\rho = \frac{-1}{\mu} \quad \text{for} \quad -1 \le \mu \le \frac{1}{(2+\tan^2\phi)^{1/2}} \quad \text{and} \quad 0 \le \phi \le \pi/4.$$

Substituting $\rho = -1/\mu$, the condition may then be expressed as

$$\begin{aligned} 0 = C_{00}^{(2)} + \sum_{n=1}^{\infty}\sum_{m=0}^{[n/4]} \Bigg[& C_{nm}^{(2)} \left(\frac{-1}{\mu}\right)^n P_n^{4m}(\mu) \\ & \times \cos(4m\phi) + C_{nm}^{(3)} \left(\frac{-1}{\mu}\right)^{-n-1} P_n^{4m}(\mu)\cos(4m\phi) \Bigg]. \end{aligned} \tag{14}$$

As mentioned above, the conditions on the top, outer side and bottom faces, equations (9), (13) and (14), constitute three components of one boundary condition. A general form of the boundary condition given by equations (9), (13) and (14) is recognized to be

$$\begin{aligned} H^{(4)}(\mu, \phi) &= C_{00}^{(2)} H^{(1)}(\mu, \phi) \\ &+ \sum_{n=1}^{\infty}\sum_{m=0}^{[n/4]} \left[C_{nm}^{(2)} H_{nm}^{(2)}(\mu, \phi) + C_{nm}^{(3)} H_{nm}^{(3)}(\mu, \phi) \right] \end{aligned} \tag{15}$$

where $H^{(1)}$, $H^{(2)}$, $H^{(3)}$ and $H^{(4)}$) are defined separately for each face [1].

To complete the solution for the coefficients $C^{(2)}$ and $C^{(3)}$, it is necessary to reduce equation (15) to a system of linear equations in terms of $C_{00}^{(1)}$ and $C_{nm}^{(1)}$ only. This is done by first expanding each of the four H functions in spherical harmonics.* By again using the orthogonality properties of spherical harmonics and through algebraic manipulation, $C_{nm}^{(2)}$ and $C_{nm}^{(3)}$ can be determined in terms of $C_{nm}^{(1)}$

$$\begin{aligned} C_{nm}^{(2)} &= C_{nm}^{(1)} \left(\frac{k_c(n+1) + k_s n}{k_c(2n+1)} \right) \\ C_{nm}^{(3)} &= \frac{C_{nm}^{(1)}(1-k_s/k_c)\, nR_a^{2n+1}}{2n+1} \end{aligned} \tag{16}$$

The spherical harmonic expansions and expressions for $C_{nm}^{(2)}$ and $C_{nm}^{(3)}$ are inserted into equation (15) along with $C_{00}^{(2)} = C_{00}^{(1)}$. With a final use of orthogonality, the following set of linear algebraic equations results

$$\begin{aligned} A1_{\gamma\eta} C_{00}^{(1)} + \sum_{n=1}^{\infty}\sum_{m=0}^{[n/4]} \Bigg\{ C_{nm}^{(1)} \Bigg[& A2_{\gamma\eta}^{nm} \left(\frac{k_c(n+1)+k_s n}{k_c(2n+1)} \right) \\ & + \frac{A3_{\gamma\eta}^{nm}(1-k_s/k_c)nR_a^{2n+1}}{2n+1} \Bigg] \Bigg\} = A4_{\gamma\eta}. \end{aligned} \tag{17}$$

Equation (17) is the defining system for the solution of the coefficients $C_{00}^{(1)}$ and $C_{nm}^{(1)}$. An upper limit is chosen for the series summation on index n. This defines the number of terms to be used in temperature solutions. Once the coefficients $C_{00}^{(1)}$ and $C_{nm}^{(1)}$ are known, it is then possible to determine the coefficients $C_{00}^{(2)}$ (equation 8c), $C_{nm}^{(2)}$ and $C_{nm}^{(3)}$ (equation 16), thus providing the coefficients necessary to evaluate the solutions for the temperature profiles within the sphere and the cube from equation (4) and (5).

3. TEMPERATURE SOLUTIONS AND POROSITY CORRECTION MODELS

The temperature distribution within a cube containing a sphere has been calculated using the solutions outlined above. Top and bottom normalized boundary conditions are 1 and 0, respectively. The sphere, representing a pore, is filled with gas with a small conductivity. Seven terms are kept in the solution series. In Fig. 3(a), the temperature field along the $\phi = 0$ face is shown. As would be expected, it is seen that the pore acts as an obstruction which causes three dimensional heat flow through the unit cell.

In Fig. 3(b), the temperatures and heat flux magnitudes on the top face are shown. As noted earlier, the temperature on this face is a boundary condition which was specified to be 1. Figure 3(b) gives an indication of error in the series solution for temperature.

To account for the presence of pores, it is customary to define a porosity correction factor which modifies the thermal conductivity of the 100% dense solid. This is usually expressed as

$$k_p = k_{100} \cdot f(p) \tag{18}$$

* For example:

$$H_{nm}^{(1)}(\mu, \phi) = A_{00}^{nm} + \sum_{\beta=1}^{\infty}\sum_{\gamma=0}^{[\alpha/4]} A_{\beta\gamma}^{nm} P_\beta^{4\gamma}(\mu)\cos(4\gamma\phi).$$

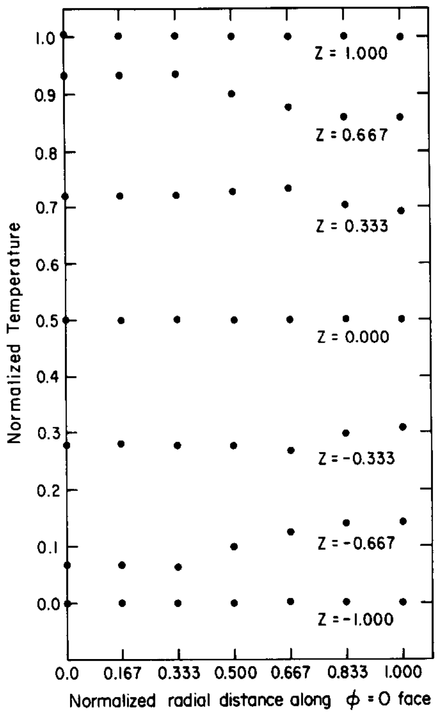

FIG. 3a. Temperature field.

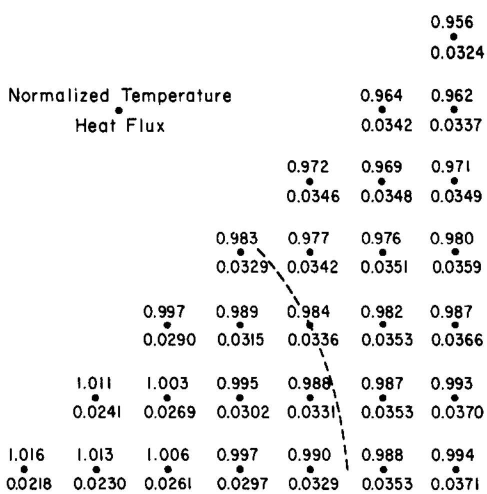

FIG. 3b. Normalized dimensions and normalized temperatures in the top face of the unit cell.

where: k_p = conductivity of the porous material; k_{100} = conductivity of 100% dense material; $f(p)$ = porosity correction for fractional porosity p.

The derivation in Section 2 can be used to find a porosity correction formula by determining the effective thermal conductivity of the unit cell from the analytical temperature solution. The heat flux crossing the top face of the unit cell is defined as

$$q''_T = -\hat{\mathbf{e}}_z \cdot [-k_c \nabla T(\rho, \mu, \phi)]\Big|_{\text{Top Face}}. \tag{19}$$

The average heat flux over the top face is determined by

$$\bar{q}''_T = \frac{\int_{TF} q''_T \, ds}{\int_{TF} ds} \tag{20}$$

where $\int_{TF} ds$ is the integral over the top face.

The effective thermal conductivity for the unit cell may now be defined as

$$k_p = |\bar{q}''_T| \cdot S/(T_T - T_B) \tag{21}$$

where k_p is the effective thermal conductivity.

By comparing the results from equation (21) to the conductivity of the 100% dense solid it is possible to derive a porosity correction formula as a function of the fractional volumetric porosity, p. The resulting expression is [3]

$$\begin{aligned} f(p) &= e^{-2.14p} \quad 0 \le p < 0.30 \\ f(p) &= 0.92 - 1.34p \quad 0.30 \le p \le 0.50. \end{aligned} \tag{22}$$

The porosity correction formula in equation (22) can be compared with other corrections obtained by several independent approaches. In the first suggestion for a porosity correction formula, the conductivity was modified by the ratio of the actual density to the maximum theoretical density [4]. This is known as Loeb's equation

$$f(p) = 1 - p. \tag{23}$$

However, Loeb's formula generally underpredicts the effect of porosity [5].

Kampf and Karsten [6] have developed an analytical correction in which a cubical pore is located in the center of a cube of a different material. Heat is assumed to flow in one direction only with no heat flow around the pore. A pore tube is oriented along the direction of the heat flow. By summing resistances to heat flow and averaging over the cross sectional area of the unit cell, the porosity correction formula obtained is

$$f(p) = 1 - p^{2/3}. \tag{24}$$

This approach has been extended by Peddicord [7] for the case of a spherical pore situated in a cubical unit cell consisting of 100% dense material. Using a similar analysis, the porosity correction formula found is

$$f(p) = 1 - \left(\frac{3\pi^{1/2}}{4}\right)^{2/3} p^{2/3}. \tag{25}$$

In both cases, equations (24) and (25), only one-dimensional heat flow was permitted.

The second approach has been to experimentally measure the conductivity of materials with known porosity. From the data, empirical expressions for correction formulas have been derived. Normally, these are assumed to be valid in the range of 0–12% porosity. The modified Loeb's formula [8, 9] is given by

$$f(p) = 1 - \alpha p \tag{26}$$

in which α usually varies between 1.7 and 2.5. The Maxwell–Eucken formula [10], another empirical expression, is given as

$$f(p) = \frac{1 - p}{1 + \beta p} \tag{27}$$

where β is between 0.5 and 1.5. Finally, in measurements of 304L stainless steel Rigimesh, 304L stainless steel sintered powders and oxygen free high-conductivity copper sintered powders, Koh and Fortini [11] have reported good agreement with the expression

$$f(p) = \frac{1 - p}{1 + 11p^2}. \tag{28}$$

In Figs. 4 and 5, the porosity correction derived in this paper [equation (22)] is compared with other porosity correction formulas. In Fig. 4, equation (22), the one-dimensional analytical corrections given by equations (24) and (25), and the Koh–Fortini expression, [equation (28)] are shown in relation to the modified Loeb equation (for $1.7 < \alpha < 2.5$). Figure 5 makes a similar comparison but for the Maxwell–Eucken correction [equation (27)]. It is noted that in both cases the analytical derivations, which do not account for multidimensional heat flow, significantly overpredict the effect of the pores. Also in both cases, the results from equation (22) are seen to fall within the band represented by the range of parameters in both empirical expressions. This is attributed to the development of equation (22) being based on multidimensional heat flow. The Koh–Fortini correction slightly underpredicts the effect of porosity in this range. In addition, an artifact of the representation in equation (28) results in a curve with slope of increasing absolute value. All other porosity correction formulas examined, both empirical and analytical, had slopes either constant or decreasing in magnitude.

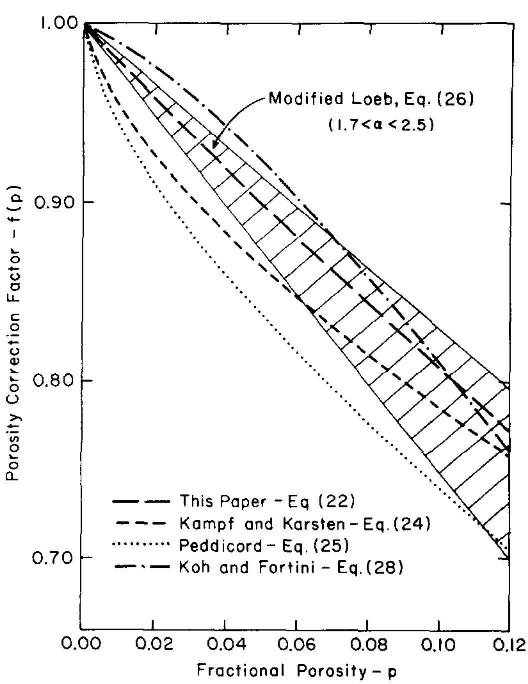

FIG. 4. Comparison of several porosity correction formulas with the modified Loeb expression.

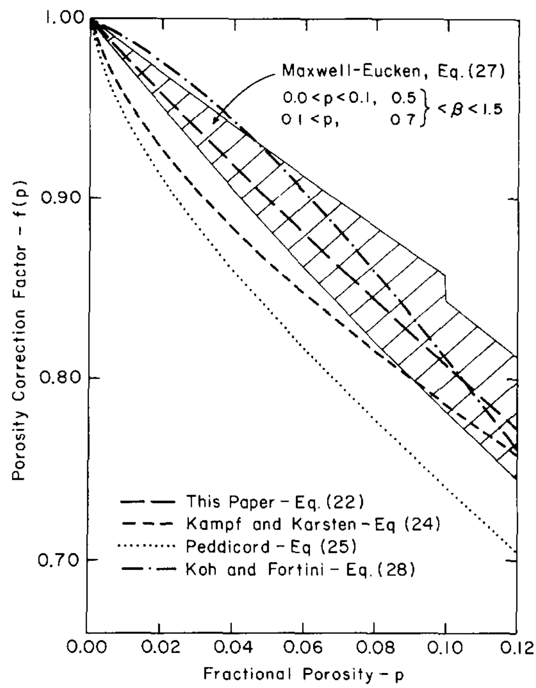

FIG. 5. Comparison of several porosity correction formulas with the Maxwell–Eucken expression.

It is often necessary to estimate the thermal conductivity in materials with porosities greater than 12%. Figure 6 compares equation (22) with the analytical correction formulas at higher porosities. It is noted that for very porous materials, the equation derived in this paper eventually approaches the one-dimensional analytical derivation for a spherical pore given by Peddicord [7]. Within the context of the unit cell representation, this indicates that as the pore becomes very large the heat cannot flow as readily around the pore and the problem is well approximated by the one-dimensional approach. At high porosities, however, equation (22) yields higher conductivities than the Koh–Fortini model, equation (28).

4. CONCLUSIONS

In this paper the three-dimensional analytical temperature solutions for a sphere occupying a cubical unit cell are presented in detail. A derived porosity correction formula generally shows good agreement with empirical expressions. In addition, the formula derived here may be used to estimate thermal conductivities at higher porosities beyond the range of validity of most empirical expressions.

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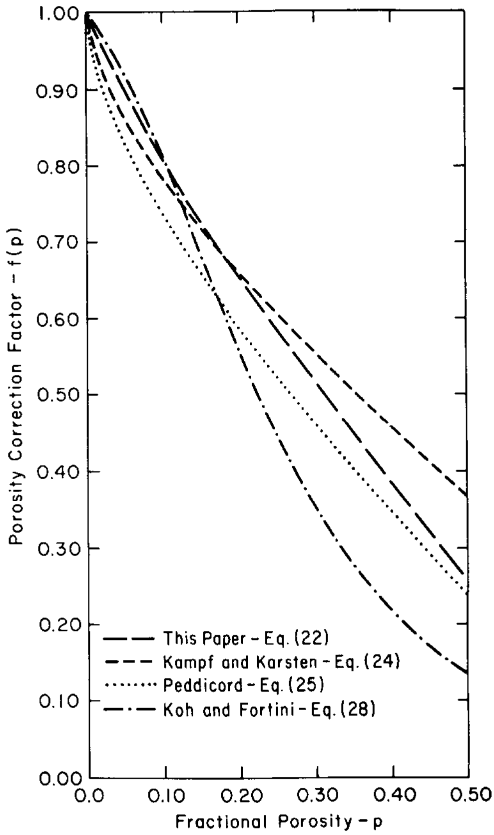

FIG. 6. Comparison of porosity correction formulas to 50% porosity.

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CONDUCTION THERMIQUE DANS DES SPHERES ASSEMBLEES EN UN ARRANGEMENT CUBIQUE REGULIER ET INFINI

Résumé — Les solutions permanentes des températures sont considérées pour un arrangement cubique de sphères. On suppose que le transfert de chaleur est par conduction et que les propriétés physiques sont constantes. Le problème est traité pour un cube unitaire contenant une sphère au centre. Du fait de la symétrie, le cube est subdivisé en un dièdre représentant la cellule unitaire. La solution analytique est obtenue pour la température en appliquant les conditions de continuité aux limites. Dans le cas où les sphères représenteraient la porosité dans un solide, on obtiendrait une correction pour la conductivité thermique sous la forme

$$f(p) = e^{-2,14p} \qquad 0 < p < 0,30$$
$$= 0,92 - 1,34p \qquad 0,30 < p < 0,50$$

où

$p =$ porosité,
$f(p) =$ facteur de correction de porosité, $f(p) \leqslant 1$.

WÄRMELEITUNG EINER UNENDLICHEN REGELMÄSSIG ANGEORDNETEN KUBISCHEN KUGELPACKUNG

Zusammenfassung — Es werden Lösungen für die Temperaturverteilung einer unendlichen kubischen Kugelpackung im stationären Fall gesucht. Der Wärmetransport erfolgt durch Leitung; die Stoffwerte werden als konstant vorausgesetzt. Das Problem wird wie ein Einheitswürfel mit einer Kugel im Mittelpunkt behandelt. Aus Symmetriegründen wird der Würfel weiter in einen Keil unterteilt, der die Einheitszelle darstellt. Durch die Anwendung von Kontinuitäts- und Randbedingungen werden die analytischen Lösungen für die Temperaturverteilung erhalten. Für den Fall, daß die Kugel einen Hohlraum innerhalb eines Festkörpers darstellt, wird ein Porositätskorrekturfaktor der Wärmeleitfähigkeit in der folgenden Form erhalten:

$$f(p) = e^{-2,14p}; \quad 0,0 < p < 0,30$$
$$= 0,92 - 1,34p; \quad 0,30 < p < 0,50.$$

Dabei ist p das Porositätsverhältnis und $f(p)$ der Porositäts-Korrekturfaktor, wobei $f(p) \leq 1$ gilt.

ТЕПЛОПРОВОДНОСТЬ БЕСКОНЕЧНОЙ КУБИЧЕСКОЙ ОБЛАСТИ, СОСТОЯЩЕЙ ИЗ СФЕР

Аннотация — Получены решения задачи о температуре для бесконечной кубической области, состоящей из сфер. Теплоперенос осуществляется теплопроводностью и свойства считаются постоянными. В задаче рассматривается единичный куб, содержащий сферу. В силу симметрии куб далее подразделяется на клин, представляющий собой единичную ячейку. Аналитические решения для температуры получены путём использования условий сплошности и граничных условий. Для случая, когда сфера представляет собой пористую ячейку в твёрдом теле, получено соотношение между пористостью и теплопроводностью, имеющее вид:

$$f(p) = e^{-2.14p} \quad 0.0 < p < 0.30$$
$$= 0.92 - 1.34p \quad 0.30 < p < 0.50$$

где p — концентрация пор

$f(p)$ — поправочный коэффициент, учитывающий пористость $f(p) \leq 1$